Technical Notes

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A Prediction of Particle Behavior via the Basset-Boussinesq-Oseen Equation

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Nomenclature

A	$=18/\sigma$, dimensionless
B	$=3/2\sigma$, dimensionless
C	$=9/[\pi^{1/2}(\sigma+0.5)]$, dimensionless
D	$=6H/[\pi d\mu_F(\sigma+0.5)]$ = normalized force, cm/s
d	= particle diameter, μm
f	= frequency of turbulent fluctuation
g	= gravitational acceleration, cm/s^2
H	= proportional to external force acting on the particle
n	= scaling or weighting factor
t	= time, s
U_F	= fluid velocity, cm/s
U_p	= particle velocity, cm/s
v_F	= kinematic viscosity of the experimental fluid
μ_F	$=\rho_F v_F$ = dynamic viscosity, Pa·s
ρ_F	= density of experimental fluid, g/cm^3
ρ_p	= density of particle material, g/cm^3
σ	$=\rho_p/\rho_F$ = particle density ratio, dimensionless
τ	$=tv_F/d^2$ = dimensionless time
Ω	= particle rotational speed

Introduction

An analytical investigation is made of the motion of marking particles in a viscous fluid during the onset of turbulence. A standard analytical technique is used to solve the Lagrangian equation of motion under the simplifying assumptions that the flow does not experience high rates of acceleration and the density of the marker material substantially is not less than the fluid density. The resultant closed-form solution for the particle motion is then examined for the case of Rayleigh-Bénard flow.

Governing Equations

The motion of a particle in a viscous fluid is governed by the Basset-Boussinesq-Oseen (BBO) equation[1-3] developed for the case of motion under gravity in a fluid at rest. The form proposed by Tchen[4] is

$$\frac{dU_p}{d\tau}+AU_p+C\int_{\tau_0}^{\tau}\frac{dU_p/d\tau'}{(\tau-\tau')^{1/2}}d\tau'=B\frac{dU_F}{d\tau}+AU_F+C\int_{\tau_0}^{\tau}\frac{dU_F/d\tau'}{(\tau-\tau')^{1/2}}d\tau'+D \tag{1}$$

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Under the assumptions mentioned, the Basset integral-type terms are negligible and, thus, the following simplified equation results:

$$\frac{dU_p}{d\tau}+AU_p=B+\frac{dU_F}{d\tau}+AU_F+D \tag{2}$$

The coefficients A and B are simple functions of the ratio of the particle to fluid densities, while the coefficient D depends on the particular external force(s) applied. For a gravitational force,

$$D=d^2g/\mu_F \tag{3}$$

and for a lift force due to particle rotation, Rubinow and Keller[5] have shown

$$D=\frac{3}{4}\frac{[d^2\Omega_p(U_p-U_F)]/v_F}{\sigma} \tag{4}$$

and Saffman[6] has represented the lift force exerted due to a velocity gradient as

$$D=\frac{81.2}{\pi}(U_F-U_p)\left(\frac{1}{v_F}\frac{dU_F}{dz}\right)^{1/2}\left(\frac{d}{4}\right)\Big/\sigma \tag{5}$$

It should be noted that, for this development, the mean fluid velocity gradient will be restricted to a uniform shear case. Thus, the BBO equation takes the form

$$\frac{dU_p}{d\tau}+\frac{18}{\sigma}U_p=\frac{1.5}{\sigma}\frac{dU_F}{d\tau}+\frac{18}{\sigma}U_F+\frac{d^2g}{\mu_F}+\frac{3}{4}\frac{d^2\Omega_p(U_p-U_F)/\sigma}{v_F}+\frac{81.2}{\pi}(U_p-U_F)\times\left(\frac{1}{v_F}\frac{dU_F}{dz}\right)^{1/2}\left(\frac{d}{4}\right)\frac{1}{\sigma} \tag{6}$$

Rearranging results in

$$\frac{dU_p}{d\tau}+U_p\left[\frac{18}{\sigma}-\frac{3d^2\Omega_p}{4v_F\sigma}-\frac{81.2}{\pi}\left(\frac{1}{v_F}\frac{dU_F}{dz}\right)^{1/2}\frac{d}{4\sigma}\right]=\frac{1.5}{\sigma}\frac{dU_F}{d\tau}+U_F\left[\frac{18}{\sigma}-\frac{3d^2\Omega_p}{4v_F\sigma}-\frac{81.2}{\pi}\times\left(\frac{1}{v_F}\frac{dU_F}{dz}\right)^{1/2}\frac{d}{4\sigma}\right]+\frac{d^2g}{\mu_F} \tag{7}$$

Note that once again the BBO equation can be cast in the following form:

$$\frac{dU_p}{d\tau}+A_1U_p=B\frac{dU_F}{d\tau}+A_1U_F+D \tag{8}$$

Defining

$$\tau^*=A_1\tau, D_1=D/A \tag{9}$$

and

$$N(\tau^*) = U_F + B\frac{dU_F}{d\tau^*} + D_1 \tag{10}$$

and using a linear differential equation technique[7] results in the following solution:

$$U_p(\tau^*) = e^{-\tau^*}\left[\int N(\tau^*)e^{\tau^*}d\tau^* + C_1\right] \tag{11}$$

where C_1 is an arbitrary constant set equal to zero, implying that the particle velocity is initially determined by the steady-state external force applied. Assuming the following general velocity field:

$$U_{Fj}(\tau^*) = \frac{\tilde{A}_j}{2}\tau^{*n}(e^{-a\tau^*} \pm e^{-b\tau^*}) \tag{12}$$

the particle velocity is given as

$$U_{pj}(\tau^*) = \frac{\tilde{A}_j}{2}\Bigg[(1-Ba)e^{-a\tau^*}\sum_{r=0}^{n}\frac{(-1)^n n!\tau^{*n-r}}{(n-r)!(-a)^{r+1}}$$

$$\pm(1-Bb)e^{-b\tau^*}\sum_{r=0}^{n}\frac{(-1)^n n!(\tau^*)^{n-r}}{(n-r)!(-b)^{r+1}}$$

$$+Bne^{-a\tau^*}\sum_{r=0}^{n}\frac{(-1)(n-1)!(\tau^*)^{n-1-r}}{(n-1-r)!(-a)^{r+1}}$$

$$\pm Bne^{-b\tau^*}\sum_{r=0}^{n}\frac{(-1)^{n-1}(n-1)!(\tau^*)^{n-1-r}}{(n-1-r)!(-b)^{r+1}} + D_1 \tag{13}$$

Application of Equation

The BBO is of particular interest to experimentalists using a laser velocimetry diagnostic technique. Flowfields that have received a considerable amount of interest from researchers attempting to understand the onset of turbulence more fully are the circular Couette and the Rayleigh-Bénard flows. In both cases, there exists a non-negligible difference in the spectral data obtained with laser velocimetry vs theoretical predictions and other more conventional techniques. Gorman et al.[8] and Gollub et al.[9] investigated the transitional phenomena and found that the auto spectra can be characterized by discrete principal components. That is,

$$S(f) \cong S_1(f_1) + (S_2(f_2) + S_3(f_3) + \ldots S_N(f_N) \tag{14}$$

or

$$f \cong n_1 f_1 + n_2 f_2 + n_3 f_3 + \ldots n f_4 \tag{15}$$

The velocity field can then be represented as

$$U_F \cong \sum_{j=1}^{N}\tilde{A}_j\cos(\tilde{\omega}_j t + \theta_j) = Re\sum_{j=1}^{N}\tilde{A}_j e^{i(\tilde{\omega}_j t + \theta_j)} \tag{16}$$

where Re implies taking the real value of the expression. The corresponding real particle velocity is calculated by Eq. (13) to be

$$U_{pj}(\tau^*) = \tilde{A}_j[(1/\omega^* - B)\sqrt{2}][\cos(\omega_j^*\tau^* + \theta_j') + D_1] \tag{17}$$

where $\theta_j' = \theta_j + 5\pi/4$.

Consider the significance of Eq. (17). First, there exists a constant phase angle shift between the particle and fluid velocities. The phase shift to this order of approximation is neither a function of the density ratios nor the viscosity of fluid. Also, the frequency of oscillation of the particle is the same as for the fluid. And, the ratio of the amplitude of the particle motion to that of the fluid motion is

$$\frac{U_{Pj}}{U_{Fj}} = \frac{1/\omega_j^* - B}{\sqrt{2}} - \frac{D_1}{\tilde{A}_j}\operatorname{sgn}\left(\frac{1}{\omega_j^* - B}\right) \tag{18}$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$. Recall also that

$$\omega_j^* = \omega_j\left(\frac{d^2}{v_F}\right)\frac{1}{A_1} \tag{19}$$

Thus,

$$\frac{U_{Pj}}{U_{Fj}} = \left(\frac{A_1}{\omega_j(d^2/v_F)} - B\right)\sqrt{2} - \frac{D_1}{\tilde{A}_j}\operatorname{sgn}\frac{1}{\omega_j^* - B} \tag{20}$$

$$\frac{U_{Pj}}{U_{Fj}} = \frac{\sqrt{2}}{\sigma}\left(\frac{18v_F}{\omega_j d^2} - \frac{3}{2}\right) - \frac{\sigma d g^2}{18\mu_F}\operatorname{sgn}(1/\omega_j^* - B) \tag{21}$$

Fiegenbaum[10] recently was able to predict the relative heights of peaks in the spectrum of a dynamical variable after a number of bifurcations have occurred. The measured peak amplitudes of Gollub et al.[9] and Libchaber et al.[11] were found to be close to the predictions of Fiegenbaum, although agreement worsened at higher frequencies. A possible explanation exists in the dynamic nature of the particle motion explained via the BBO equation, and the solution documented in this investigation. Equation (17) asserts that the laser Doppler measurement of the spectra should be smaller in magnitude at a given discrete frequency than predicted by fluid theory and hot-wire anemometry data. This assertion has, in fact, been shown to be true, as has the worsening trend at higher frequencies.

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